FINITE ELEMENT EVALUATION OF THE DYNAMIC RESPONSE OF BEAMS UNDER UNIFORMLY DISTRIBUTED MOVING LOADS

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ABSTRACT

A detailed analysis of vibration of beams under uniformly distributed moving loads using finite element method is carried out. The material properties, throughout the length of the structures under consideration are assumed to be prismatic. The weak form of the equation describing the vibration of beams is obtained using Galerkin's Weighted Residual Method (GWRM) while the elements stiffness, mass, and centripetal acceleration matrices as well as the load vectors were derived. Newmark's integration method is used to obtain the dynamic response of beams under uniformly distributed moving loads. Numerical examples are presented to show the effects of :(i) velocity of the moving load; (ii) load's length on the dynamic response of beam under uniformly distributed moving loads.

INTRODUCTION

As a result of ever increasing in global increases in the demand for the transformations in transport sectors, most especially, in railway system, pavements, carriage ways and many more. The need for provocative research involving structural members, such as beams under the influences of the moving loads.

Since the middle of the last century, when railway construction began, the problem of vibration of bridges under travelling loads has interested engineers, mathematicians and other scientists. Contributions to the solution of this problem were made by Sir George Stokes (1849), Robert Wills (1849) and many others. Stephen Timoshenko (1927) considered the case of pulsating load passing on the bridge. Sir Charles Inglis (1934), in his systematic analysis of train crossing a bridge took into account many important factors, such as the effects of the moving loads, the influence of damping and the spring suspension of the locomotive.

For the case of a concentrated force moving with a constant velocity along a beam, neglecting damping forces Timoshenko (1953) found a solution, and presented an expression for the critical velocity. Stanisic and Hardin (1968) determined the dynamic behaviour of a simply supported beam carrying a moving mass which is interesting enough, but their method is not easily applicable to different boundary conditions. A comprehensive treatment of the subject of vibrations of structures to loads, which contains a large number of related cases, is that of Fryba (1972).

Akin and Mofid (1989) presented an analyti-

cal numerical method that can be used to determine the dynamic behaviour of beams with different boundary conditions carrying a concentrated moving mass. The problem of dynamic behaviour of an elastic beam subjected to a moving concentrated mass was also studied by Sadiku and Leipholz (1989). Gbadeyan and Oni (1995) presented a more versatile technique which can be used to determine the dynamic behaviour of beams having arbitrary end supports. Esmailzadeh and Ghorashi (1995) investigated the problem of vibration analysis of beams due to partially distributed, uniformly moving masses.

Michaltos, Sophianopoulos and Kounadis (1996), studied the effect of the mass of a moving load on the dynamic response of a simply supported beam. Some interesting results were obtained. A detailed analysis of the effect of centripetal and coriolis forces on the dynamic response of light (steel) bridges under moving loads was also carried out by Michaltos and Kounadis (2001). Gbadeyan *et al.* (2002) investigated the dynamic response of beams subjected to uniform partially distributed moving masses in which the inertia of the load was taken into consideration. It is remarked at this juncture, that, in all the works aforementioned so far, analytical methods were employed.

In this paper, the finite element method was employed to obtain the dynamic response of beams under uniformly distributed moving loads. The finite element model of the problem was obtained by applying Galerkin's Weighted Residual Method (GWRM), the responses were obtained using Newmark's integration method (1959) with the aid of a computer code written in Visual Basic pro-

Problem Formulation

For moving load problem, such as a train moving on a bridge, the Euler-Bernoulli equation for beam bending is (Esmailzadeh *et al.,* 1995):

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho\frac{\partial^2 y(x,t)}{\partial t^2} = q(x,t)$$

(1)

where $y(x,t)$ is the transverse displacement of the beam, ρ is the mass density per volume, EI is the beam rigidity, $q(x,t)$ is the externally applied pressure loading, t and x indicate time and the spatial axis along the beam axis.

The associated boundary conditions are:

$$y(0,t) = y(l,t) = 0$$

$$\frac{\partial^2 y(x,t)}{\partial x^2}\Big|_{x=0} = \frac{\partial^2 y(x,t)}{\partial x^2}\Big|_{x=l} = 0$$

(2)

For moving load problem,

$$q(x,t) = \frac{1}{\varepsilon}[-pg - p(\frac{\partial^2 y}{\partial x^2} + 2v\frac{\partial^2 y}{\partial x\partial t} + v^2\frac{\partial^2 y}{\partial x^2})][H(x - \xi + \frac{\varepsilon}{2}) - H(x - \xi - \frac{\varepsilon}{2})]$$

(3)

Using (3) in (1), we have:

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\rho\frac{\partial^2 y(x,t)}{\partial t^2}=$$

$$\frac{1}{\varepsilon}[-pg-p(\frac{\partial^2 y}{\partial x^2}+2v\frac{\partial^2 y}{\partial x\partial t}+v^2\frac{\partial^2 y}{\partial x^2}][H(x-\xi+\varepsilon/2)-H(x-\xi-\varepsilon/2)] \quad (4)$$

In order to solve Equation (4) using finite element method, since the closed-form solution of the problem is either impossible or very difficult using analytical approach, we use Galerkin's weighted Residual Method (GWRM) to obtain the weak formulation of the problem.

The weak formulation of the Beam Equation

The weak formulation of Equation (4) according to Kwon *et al., (*1996) and Reddy (1993) is

$$\int_0^l\{\quad EI\frac{\partial^4 y(x,t)}{\partial x^4}+\rho\frac{\partial^2 y(x,t)}{\partial t^2}\quad\}Rdx=$$

$$\int_0^l\{\quad\frac{1}{\varepsilon}[-pg-p(\frac{\partial^2 y}{\partial x^2}+2v\frac{\partial^2 y}{\partial x\partial t}+v^2\frac{\partial^2 y}{\partial x^2}][H(x-\xi+\varepsilon/2)-H(x-\xi-\varepsilon/2)]\quad\}Rdx$$

$$(5)$$

where, R is the Galerkin's weight or test function.
Rearranging Equation (5), we obtain:

$$EI\int_0^l\frac{\partial^4 y}{\partial x^4}Rdx+\rho\int_0^l\frac{\partial^2 y}{\partial t^2}Rdx=$$

$$-pg/\varepsilon\int_0^l R[H(x-\xi+\varepsilon/2)-H(x-\xi-\varepsilon/2)]dx$$

$$-p/\varepsilon\int_0^l\frac{\partial^2 y}{\partial t^2}R[H(x-\xi+\varepsilon/2)-H(x-\xi-\varepsilon/2)]dx$$

$$-2pv/\varepsilon\int_0^l\frac{\partial^2 y}{\partial x\partial t}R[H(x-\xi+\varepsilon/2)-H(x-\xi-\varepsilon/2)]dx \quad\Biggr\} \quad (6)$$

$$-pv^2/\varepsilon\int_0^l\frac{\partial^2 y}{\partial x^2}R[H(x-\xi+\varepsilon/2)-H(x-\xi-\varepsilon/2)]dx$$

Integrating twice by parts the 1st term on the left-hand side of (6) , we have:

$$EI\int_0^l\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 R}{\partial x^2}dx+[\vartheta R-\lambda\frac{\partial R}{\partial x}]_0^l+\rho\int_0^l\frac{\partial^2 y}{\partial t^2}Rdx=$$

$$- \frac{pg}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} R dx - \frac{P}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \frac{\partial^2 y}{\partial t^2} R dx - \frac{2pv}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \frac{\partial^2 y}{\partial x \partial t} R dx - \frac{pv^2}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \frac{\partial^2 y}{\partial x^2} R dx \qquad (7)$$

where:

$$\vartheta = EI(\frac{\partial^3 y}{\partial x^3}), the - shearforce$$

$$\lambda = EI(\frac{\partial^2 y}{\partial x^2}), the - bending - moment$$

The approximation of the integral (Dada, 2003)

$$\int_0^l f(x)[H(x-\xi+\varepsilon/2) - H(x-\xi-\varepsilon/2)]dx = \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} f(x)dx \qquad (8)$$

Finite Element Model of the Beam Equation

The standard mathematical discretization (Kwon *et al.*, 1996) of beam Equation (7) into a number of finite elements yields;

$$\sum_{i=1}^{n}\{ \quad EI\int_{\Omega} \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 R}{\partial x^2} dx + [\vartheta R - \lambda \frac{\partial R}{\partial x}]_0^l + \rho \int_{\Omega} \frac{\partial^2 y}{\partial t^2} R dx =$$

$$- \frac{pg}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} R dx - \frac{P}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \frac{\partial^2 y}{\partial t^2} R dx - \frac{2pv}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \frac{\partial^2 y}{\partial x \partial t} R dx - \frac{pv^2}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \frac{\partial^2 y}{\partial x^2} R dx \}$$

$$(9)$$

here: $\Omega = l_e$, the domain of the beam element.

Finally, the finite element form of (9) is:

$$[K]\{y\} + [C]\{\dot{y}\} + [M]\{\ddot{y}\} = \{F\} \qquad (10)$$

where:

$$[K] = \sum_{i=1}^{n}\{ \quad EI\int_{\Omega} \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 R}{\partial x^2} dx \quad + \quad \frac{pv^2}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \frac{\partial^2 y}{\partial x^2} R dx \} \qquad (11a)$$

$$[M] = \sum_{i=1}^{n}\{ \rho\int_{0}^{l-} \frac{\partial^2 y}{\partial t^2} R dx + \frac{P}{\varepsilon} \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \frac{\partial^2 y}{\partial t^2} R dx \} \qquad (11b)$$

$$[C] = \sum_{i=1}^{n} \left\{ \left. 2pv \middle/ \varepsilon \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} \frac{\partial^2 y}{\partial x \partial t} R dx \right\} \right. \tag{11c}$$

$$\{F\} = \sum_{i=1}^{n} \left\{ \left. -pg \middle/ \varepsilon \int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} R dx + Q^e \right\} \right. \tag{11d}$$

By using Hermittian interpolation functions (Cheung *et al.,*1978; Reddy, 1993)

$$H^T = \left[1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \quad x - \frac{2x^2}{l} + \frac{x^3}{l^2} \quad \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \quad -\frac{x^2}{l} + \frac{x^3}{l^2} \right] \tag{11e}$$

to interpolate the transverse displacement in the above equations, therefore, from equation (11a), we have, for a single element, the system stiffness matrix:

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \tag{12}$$

where:

$$K_{11} = \frac{12EI}{l^3} + \frac{pv^2}{\varepsilon} \left[-6\eta + \frac{6\eta^3}{l^2} - \frac{3\eta^4}{l^3} + \frac{6\eta^2}{l^3} - \frac{9\eta^4}{l^5} + \frac{24\eta^5}{5l^6} \right]$$

$$-\frac{pv^2}{\varepsilon} \left[-6\mu + \frac{6\mu^3}{l^2} - \frac{3\mu^4}{l^3} + \frac{6\mu^2}{l^3} - \frac{9\mu^4}{l^5} + \frac{24\mu^5}{5l^6} \right]$$

$$K_{12} = \frac{6EI}{l^2} + \frac{pv^2}{\varepsilon} \left[-3\eta^2 + \frac{4\eta^3}{l} - \frac{3\eta^4}{2l^2} + \frac{4\eta^3}{l^3} - \frac{6\eta^4}{l^4} + \frac{12\eta^5}{5l^5} \right]$$

$$-\frac{pv^2}{\varepsilon} \left[-3\mu^2 + \frac{4\mu^3}{l} - \frac{3\mu^4}{2l^2} + \frac{4\mu^3}{l^3} - \frac{6\mu^4}{l^4} + \frac{12\mu^5}{5l^5} \right]$$

$$K_{13} = \frac{-12EI}{l^3} + \frac{pv^2}{\varepsilon} \left[-\frac{6\eta^3}{l^2} + \frac{3\eta^4}{l^3} + \frac{9\eta^4}{l^5} - \frac{24\eta^5}{5l^6} \right]$$

$$-\frac{pv^2}{\varepsilon} \left[-\frac{6\mu^3}{l^2} + \frac{3\mu^4}{l^3} + \frac{9\mu^4}{l^5} - \frac{24\mu^5}{5l^6} \right]$$

$$K_{14} = \frac{6EI}{l^2} + \frac{pv^2}{\varepsilon} \left[\frac{2\eta^3}{l} - \frac{3\eta^4}{2l^2} - \frac{3\eta^4}{l^4} + \frac{12\eta^5}{5l^5} \right]$$

$$- \frac{pv^2}{\varepsilon} \left[\frac{2\mu^3}{l} - \frac{3\mu^4}{2l^2} - \frac{3\mu^4}{l^4} + \frac{12\mu^5}{5l^5} \right]$$

$$K_{21} = \frac{6EI}{l^2} + \frac{pv^2}{\varepsilon} \left[-\frac{4\eta}{l} + \frac{4\eta^3}{l^3} - \frac{13\eta^4}{2l^4} + \frac{3\eta^2}{l^2} + \frac{12\eta^5}{5l^5} \right]$$

$$- \frac{pv^2}{\varepsilon} \left[-\frac{4\mu}{l} + \frac{4\mu^3}{l^3} - \frac{13\mu^4}{2l^4} + \frac{3\mu^2}{l^2} + \frac{12\mu^5}{5l^5} \right]$$

$$K_{22} = \frac{4EI}{l} + \frac{pv^2}{\varepsilon} \left[-\frac{2\eta^2}{l} + \frac{4\eta^3}{3l^3} - \frac{4\eta^4}{l^3} + \frac{6\eta^5}{5l^4} \right]$$

$$- \frac{pv^2}{\varepsilon} \left[-\frac{2\mu^2}{l} + \frac{4\mu^3}{3l^3} - \frac{4\mu^4}{l^3} + \frac{6\mu^5}{5l^4} \right]$$

$$K_{23} = -\frac{6EI}{l^2} + \frac{pv^2}{\varepsilon} \left[-\frac{4\eta^3}{l^3} + \frac{13\eta^4}{2l^4} - \frac{12\eta^5}{5l^5} \right] - \frac{pv^2}{\varepsilon} \left[-\frac{4\mu^3}{l^3} + \frac{13\mu^4}{2l^4} - \frac{12\mu^5}{5l^5} \right]$$

$$K_{24} = \frac{2EI}{l} + \frac{pv^2}{\varepsilon} \left[\frac{4\eta^3}{3l^2} - \frac{5\eta^4}{2l^3} + \frac{6\eta^5}{5l^4} \right] - \frac{pv^2}{\varepsilon} \left[\frac{4\mu^3}{3l^2} - \frac{5\mu^4}{2l^3} + \frac{6\mu^5}{5l^4} \right]$$

$$K_{31} = -\frac{12EI}{l^3} + \frac{pv^2}{\varepsilon} \left[\frac{6\eta}{l^2} - \frac{6\eta^3}{l^4} + \frac{12\eta^4}{l^5} - \frac{6\eta^2}{l^3} - \frac{24\eta^5}{5l^6} \right]$$

$$- \frac{pv^2}{\varepsilon} \left[\frac{6\mu}{l^2} - \frac{6\mu^3}{l^4} + \frac{12\mu^4}{l^5} - \frac{6\mu^2}{l^3} - \frac{24\mu^5}{5l^6} \right]$$

$$K_{32} = -\frac{6EI}{l^2} + \frac{pv^2}{\varepsilon} \left[\frac{3\eta^2}{l^2} - \frac{8\eta^3}{l^3} + \frac{15\eta^4}{2l^4} - \frac{12\eta^5}{5l^5} \right]$$

$$- \frac{pv^2}{\varepsilon} \left[\frac{3\mu^2}{l^2} - \frac{8\mu^3}{l^3} + \frac{15\mu^4}{2l^4} - \frac{12\mu^5}{5l^5} \right]$$

$$K_{33} = \frac{12EI}{l^3} + \frac{pv^2}{\varepsilon} \left[\frac{6\eta^3}{l^3} - \frac{12\eta^4}{l^5} + \frac{24\eta^5}{5l^6} \right] - \frac{pv^2}{\varepsilon} \left[\frac{6\mu^3}{l^3} - \frac{12\mu^4}{l^5} + \frac{24\mu^5}{5l^6} \right]$$

$$K_{34} = -\frac{6EI}{l^2} + \frac{pv^2}{\varepsilon} \left[-\frac{2\eta^3}{l^3} + \frac{9\eta^4}{2l^4} - \frac{12\eta^5}{5l^5} \right] - \frac{pv^2}{\varepsilon} \left[-\frac{2\mu^3}{l^3} + \frac{9\mu^4}{2l^4} - \frac{12\mu^5}{5l^5} \right]$$

$$K_{41} = \frac{6EI}{l^2} + \frac{pv^2}{\varepsilon}\left[-\frac{2\eta}{l} + \frac{2\eta^3}{l^3} - \frac{11\eta^4}{2l^4} + \frac{3\eta^2}{l^2} + \frac{12\eta^5}{5l^5} \right]$$

$$-\frac{pv^2}{\varepsilon}\left[-\frac{2\mu}{l} + \frac{2\mu^3}{l^3} - \frac{11\mu^4}{2l^4} + \frac{3\mu^2}{l^2} + \frac{12\mu^5}{5l^5} \right]$$

$$K_{42} = \frac{2EI}{l} + \frac{pv^2}{\varepsilon}\left[-\frac{\eta^2}{l} + \frac{10\eta^3}{3l^2} - \frac{7\eta^4}{2l^3} + \frac{6\eta^5}{5l^4} \right]$$

$$-\frac{pv^2}{\varepsilon}\left[-\frac{\mu^2}{l} + \frac{10\mu^3}{3l^2} - \frac{7\mu^4}{2l^3} + \frac{6\mu^5}{5l^4} \right]$$

$$K_{43} = -\frac{6EI}{l^2} + \frac{pv^2}{\varepsilon}\left[-\frac{2\eta^3}{l^3} + \frac{11\eta^4}{2l^4} - \frac{12\eta^5}{5l^5} \right] - \frac{pv^2}{\varepsilon}\left[-\frac{2\mu^3}{l^3} + \frac{11\mu^4}{2l^4} - \frac{12\mu^5}{5l^5} \right]$$

$$K_{44} = \frac{4EI}{l} + \frac{pv^2}{\varepsilon}\left[-\frac{2\eta^3}{3l^2} - \frac{2\eta^4}{l^3} + \frac{6\eta^5}{5l^4} \right] - \frac{pv^2}{\varepsilon}\left[-\frac{2\mu^3}{3l^2} - \frac{2\mu^4}{l^3} + \frac{6\mu^5}{5l^4} \right]$$

where: $\eta = \xi + \varepsilon/2$ and $\mu = \xi - \varepsilon/2$

Similarly, interpolating Equations (11b) to (11d) using the same shape function yields the consistent element mass matrix $[M]$, the centripetal acceleration matrix $[C]$ and the element force vector $\{F\}$ respectively. In this paper, the consistent mass matrix [9] is used for the analysis carried out.

The specification of Q^e in Equation (11d) depends on the associated boundary conditions for a particular problem. However, in this paper, the emphasis is on simply supported beams with a little comparison with cantilever beams.

Numerical Examples

In this paper, a $10m$ long two-node simply supported structural beam element was modeled (Fig.1a) to illustrate the above procedure. The beam element is discretized into 6 uniform elements (Fig.1b) with homogeneous materials.

In addition, the mass density per beam length $\rho = 7800 kg/m^3$, the flexural rigidity $EI = 2.7728 \times 10^5 \, N/m^2$, the beam cross sectional area $A = 1m^2$, the lateral load $P = 1000N$, the velocity of the moving load $V = 8m/s$, and the load's length $\varepsilon = 0.5m$. In order to obtain the effect of the velocity and load's length on the dynamic response of beam elements to moving loads, various values for both parameters were used:

(a) **Effects of velocity on the dynamic response of the beam**: The effect of increasing in velocity on the dynamic

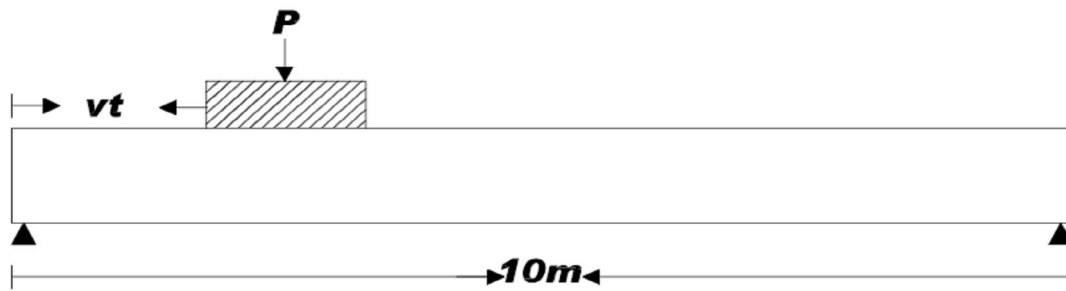**Figure (1a): A uniform beam under moving load**

Figure (1b): A discretized uniform beam

response of a simply supported beam under distributed moving load is shown in Figure 2. It shows that for the initial velocity V_0 smaller than a certain value, denoted by $V_0^/$, the value of the deflections (y) increases with increasing in velocity. However, for $V_0 > V_0^/$, the foregoing trend just reverses, the critical value of the initial velocity for this problem is $V_0^/ = 15m/s$, while the reverse case is shown in figure 3. The implication is that after exceeding the critical value of the velocity, the deflections decreases as the velocity increases.

(b) Effects of load's length: In order to investigate the influence of the load's length on the dynamic response of a simply supported beam having the same properties as those of the one in Figure 3, but with $\varepsilon = 0.5$, $\varepsilon = 0.7$, $\varepsilon = 0.9$, respectively, were studied. This shows that the deflections (y) increases with increasing in load's length as described in Figure4. Finally, the findings in (a) shows similar pattern in the dynamic responses with the one obtained in (4, 7), while (b) is an extension of the results obtained in (4, 7, 10, 11).
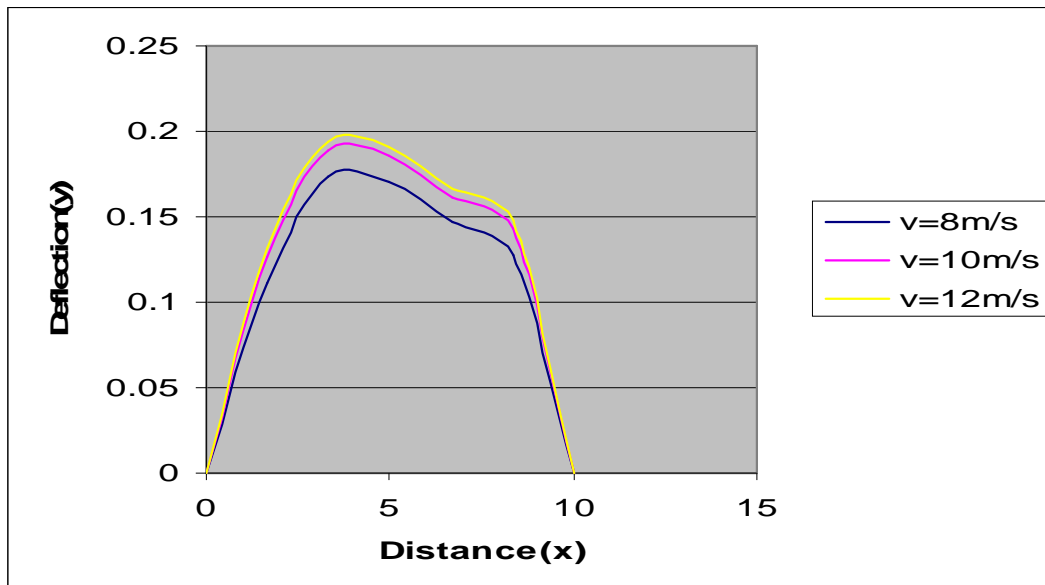
Figure 2: Effect of increasing velocity on the dynamic responses of beams under moving loads
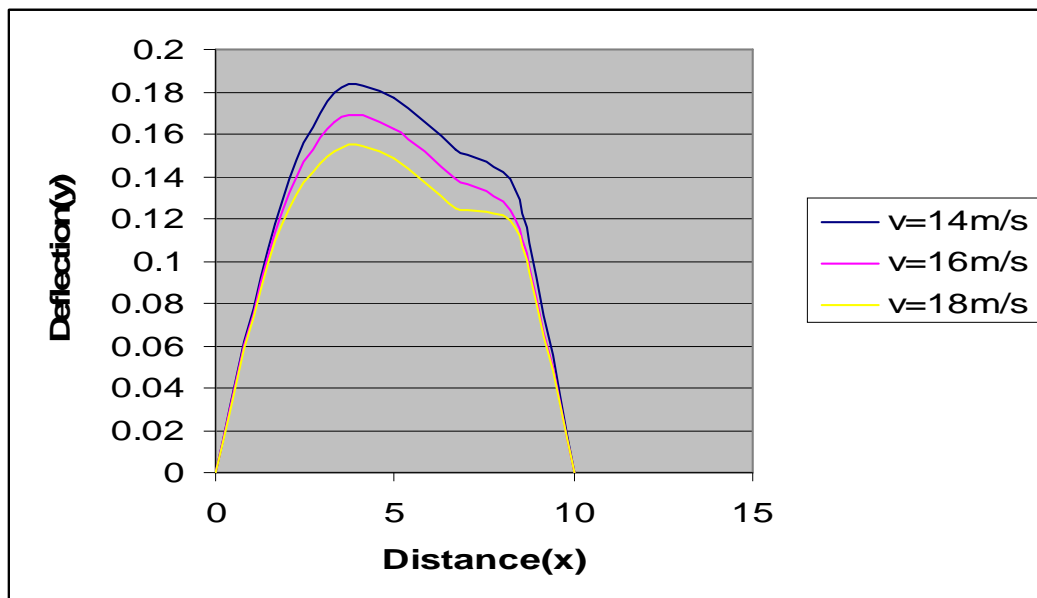

Figure 3: Effect of exceeding the critical value of the velocity on the dynamic responses of beams under moving loads
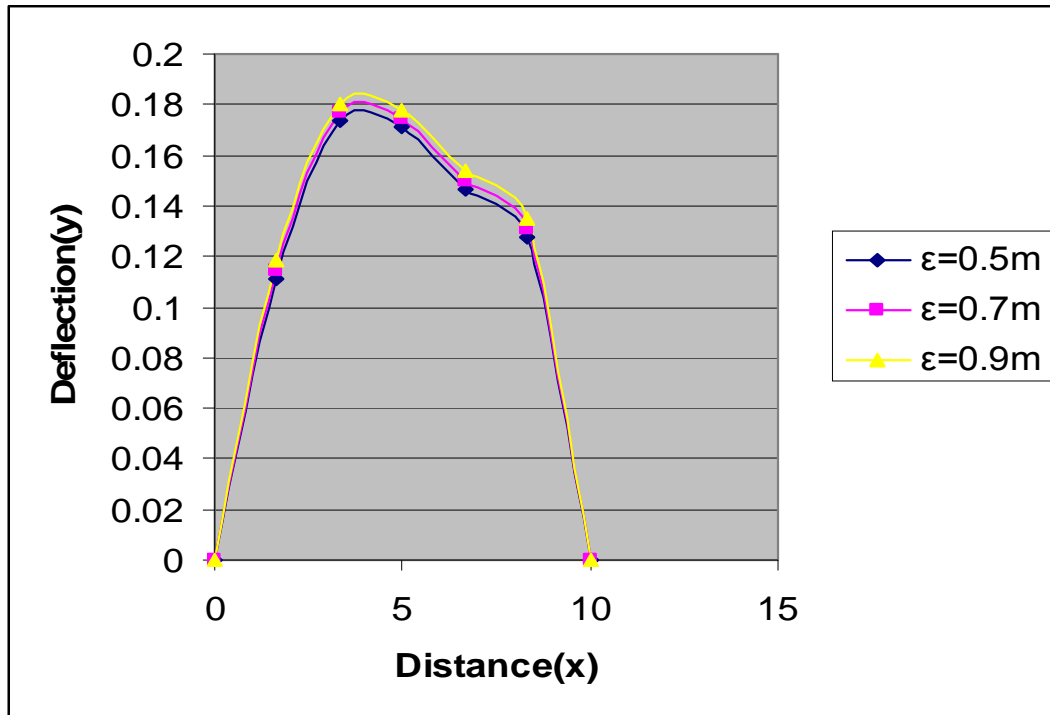
Figure 4: Effect of increases in load's length on the dynamic responses of beams under moving loads

CONCLUSION

In this paper, a detailed analysis of vibration of beams under uniformly distributed moving loads has been studied. The dynamic response of beams subjected to uniformly distributed moving loads using finite element method, incorporating the Newmark's β numerical technique for the evaluation of the resulted equations in order to obtain the effects of velocity of the moving load and load's length on the response of beams. It is concluded that the velocity of the moving loads and load's length have significant effects on the dynamic response of beams under uniformly distributed moving loads. The results obtained, for the effects of velocity of the moving loads, are in agreement with those in the existing literatures, while for the load's length, the response amplitude is directly proportional to the values of the load's length.

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